A tropical Nullstellensatz

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Overview

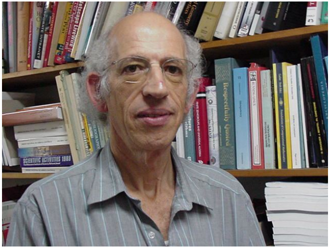
Introduction to tropical mathematics

The foundations of tropical mathematics were laid in the 1960s by Cuninghame-Green and Vorobyev. Early developments were due to:

- Imre Simon (1976)
- the Russian school led by Maslov - 80s
- Oleg Viro - 80s
- the French Max-plus working group - 80s
- Tropical geometry: Mikhalkin, Itenberg, Sturmfels

Introduction to tropical mathematics

They were named in honour of Imre Simon who lived and worked in Sao Paulo.



Introduction to tropical mathematics

In tropical geometry, we consider the max-plus semi-field $\mathbb{R}_{max}$, the set $\mathbb{R} \cup \{-\infty\}$ with the two operations, $x \oplus y = max\{x, y\}$ and $x \odot y = x + y$.

For instance $3 \oplus 5 = 5$, $3 \odot 5 = 3 + 5 = 8$, $2^{\odot 3} = 2 \times 3 = 6$.

We sometimes use quotation marks to denote operations in the tropical world: "$3 + 3 = 3$"

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Little exercise:

"$\sqrt{-2}$"?

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$$"\sqrt{-2}" = -1 \text{ as } "(-1)^2" = -1 - 1 = -2$$

We can also consider the min-plus semi-field.

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Tropical monomial in variables $\overrightarrow{x} = (x_1, ..., x_n)$:

$m(\overrightarrow{x}) = c \odot x_1^{\odot i_1} \odot ... \odot x_n^{\odot i_n}$ where all exponents are non-negative integers. In classical terms, $m(\overrightarrow{x}) = c + \langle i, \overrightarrow{x} \rangle$ is an affine function with non-negative integer slope.

For example $m_1(x, y) = 3 \odot x \odot y^{\odot 2} = 3 + x + 2y$

Tropical polynomial: $"\sum_{i=1}^{n} M_i(\overrightarrow{x})" = max_i M_i(\overrightarrow{x})$
each $M_i(\overrightarrow{x})$ is a tropical monomial in variables $\overrightarrow{x} = (x_1, ..., x_n)$

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Definition

The degree of a tropical monomial is the sum of its exponents. The degree of a tropical polynomial f, $deg(f)$, is the maximal degree of its monomials.

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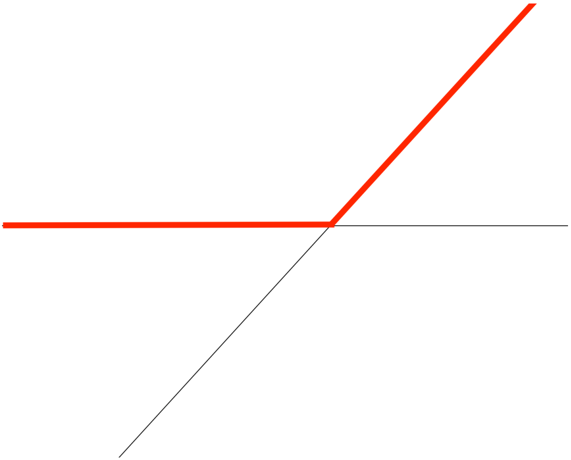
In one variable: let us look at $P_1(x) = "0 + x"$



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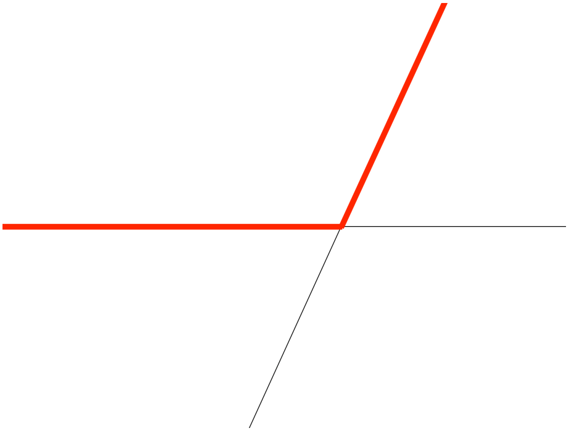
...at $P_2(x) = "0 + x^2"$



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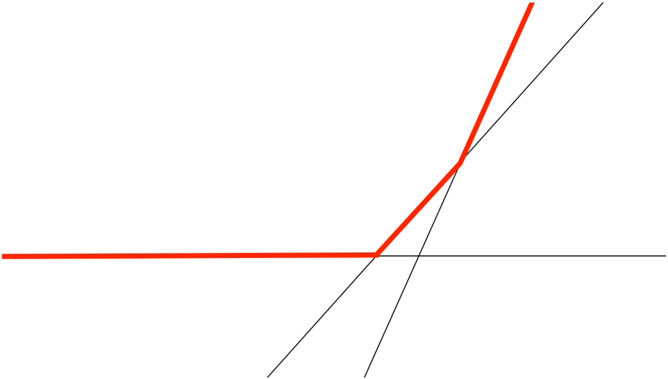
... and at $P_3(x) = "0 + x + (-1)x^2" = "(-1)(x+0)(x+1)"$



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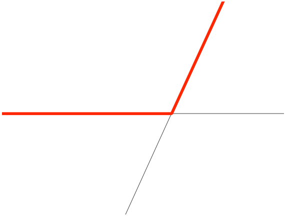
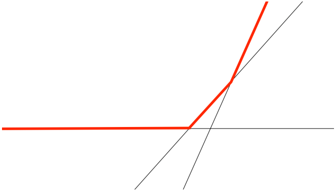
In one variable:

- Tropical roots of the polynomial $P(x)$: points x_0 at which the graph $P(x)$ has a corner at x_0.
- The difference in the slopes of the two pieces adjacent to a corner gives the order of the corresponding root.

$P_2(x) = "0 + x^2"$



$P_3(x) = "0 + x + (-1)x^2"$



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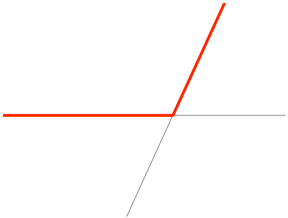
We can factorise: if a is a root of $P(x)$ of degree b, $P(x) = "(x+a)^b Q(x)"$. Note that we factorize by $"(x+a)"$ and not by $(x-a)$ as we would usually.

$$P_2(x) = "0 + x^2" = "(x+0)(x+0)" = 2max\{x+0\}$$



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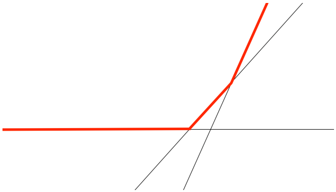
We can factorise: if a is a root of $P(x)$ of degree b, $P(x) =" (x+a)^b Q(x)"$.

$P_3(x) =" 0 + x + (-1)x^2" =" (-1)(x+0)(x+1)" = -1 + max\{x,0\} + max\{x,1\}$



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The tropical semi-field is algebraically closed. In other words every tropical polynomial of degree d has exactly d roots when counted with multiplicities.

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In two variables: $P(x,y) =" \sum_{i,j} a_{i,j} x^i y^j " = max_{i,j}(a_{i,j} + ix + jy)$.

Introduction to tropical mathematics

The tropical curve C defined by $P(x, y)$ is defined as the corner locus of this function (ie all the points for which the maximum of $P(x, y)$ is attained at least twice).

C is the set of points (x_0, y_0) of $\mathbb{R}^2$ such that there exists pairs $(i, j) \neq (k, l)$ satisfying $P(x_0, y_0) = a_{i,j} + ix_0 + jy_0 = a_{k,l} + kx_0 + ly_0$

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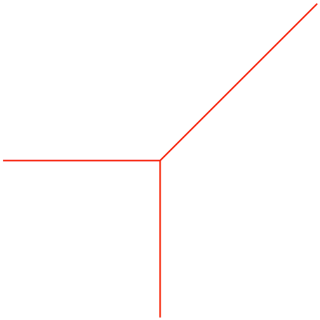
Example: $P(x, y) =$"$0 + x + y$"$= max(0, x, y)$. We look at $x = 0 \geq y$, $y = 0 \geq x$, $x = y \geq 0$:



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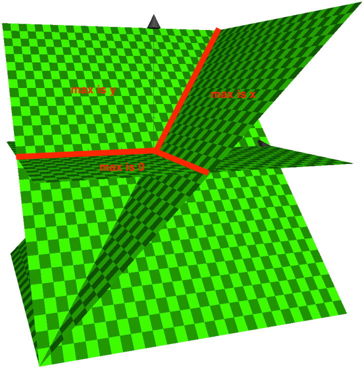

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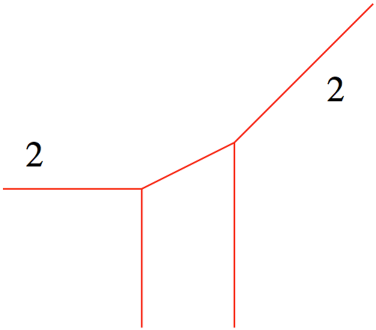
Example: $P(x,y) = "0 + x + y + y^2 + (-1)x^2"$. We proceed as above: maximum reached twice.



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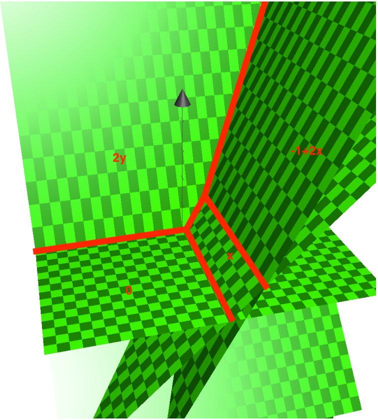

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Definition

The weight w_e of an edge e of C is defined to be the maximum of the greatest common divisor of the numbers $|i-k|$ and $|j-l|$ for all pairs (i,j) and (k,l) which correspond to this edge. [BS14]

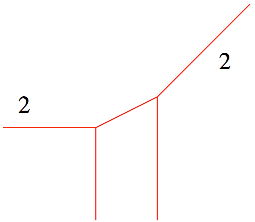
That is to say $w_e = max_{M_e}(gcd(|i-k|,|j-l|)$ where

$$M_e = \{(i,j),(k,l)|\forall x_0 \in e, P(x_0,y_0) = a_{i,j}+ix_0+jy_0 = a_{k,l}+kx_0+ly_0\}$$

Introduction to tropical mathematics

Example: for $P(x,y) = "0 + x + y + y^2 + (-1)x^2"$:

- the edge corresponding to $2y = 0 \geq y, x, -1+2x$ is associated to $M_e = \{(0,2),(0,0),(0,1)\}$
- the edge corresponding to $x = 0 \geq y, 2y, -1+2x$ is associated to $M_e = \{(1,0),(0,0)\}$
- the edge corresponding to $2y = -1+2x \geq x, 0$ is associated to $M_e = \{(0,2),(2,0)\}$



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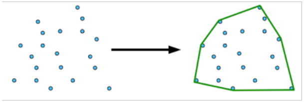
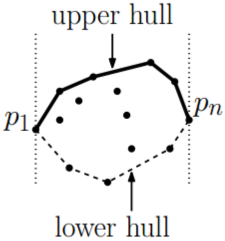
Definition

The convex hull of a set X of points in the Euclidean plane or in a Euclidean space is the smallest convex set that contains X.





Introduction to tropical mathematics

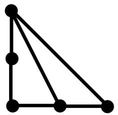
There is an algorithm to draw a tropical curve [I08]:

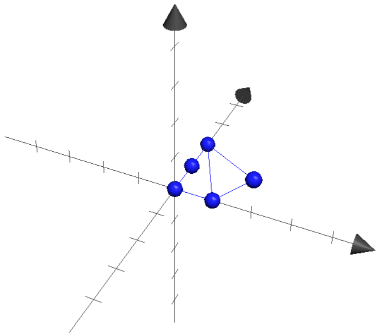
- To each term $a_{i,j} \odot x^i \odot y^j$ in the polynomial, associate the point $(i, j, -a_{i,j})$ in $\mathbb{R}^3$
- Compute the convex hull of these points in $\mathbb{R}^3$
- Now project the lower envelope of that convex hull into the plane under the map $\mathbb{R}^3 \to \mathbb{R}^2$, $(x, y, z) \mapsto (x, y)$
- The image is a planar convex polygon together with a distinguished subdivision Δ into smaller polygons
- The tropical curve is the dual graph to this subdivision

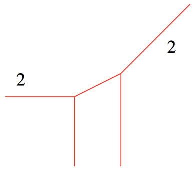
Introduction to tropical mathematics

$$P(x,y) = \text{"}0 + x + y + y^2 + (-1)x^2\text{"} = \text{"}0 + 0x + 0y + 0y^2 + (-1)x^2\text{"}$$

So plot the points $(0,0,0)$, $(0,1,0)$, $(0,0,1)$, $(0,0,2)$, $(1,2,0)$.







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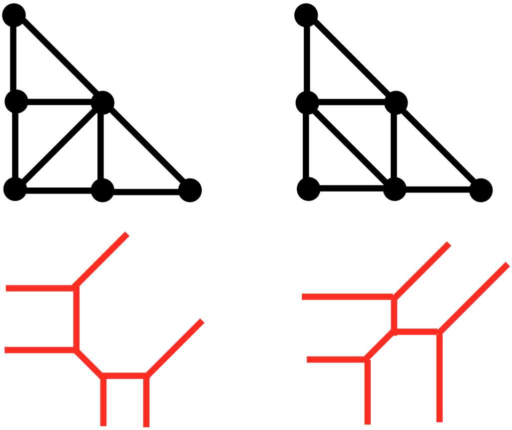
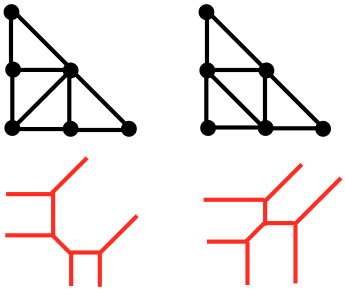


Introduction to tropical mathematics

Theorem

An edge e of a tropical curve has weight w if and only if $Card(\Delta_e \cap \mathbb{Z}^2) = w + 1$. [BS14]

For instance, for the following tropical curves, all edges have weight 1.



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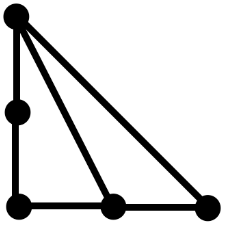
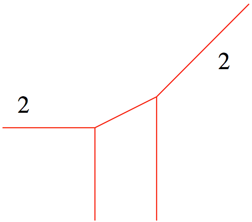
For $P(x, y) = "0 + x + y + y^2 + (-1)x^2"$, we get the following Newton's polygon:





We therefore have two edges with weight 2.

Introduction to tropical mathematics

Let's try another example:
$P(x,y) = "3 + 5x^2y + x" = "3 + 5x^2y + 0x"$
So plot the points $(0,0,-3)$, $(2,1,-5)$, $(1,0,0)$.

Introduction to tropical mathematics

Let (i, j) be a vector with integer coordinates in $\mathbb{R}^2$. Let Λ_p be a finite collection of points with integer coordinates in $\mathbb{R}^2$.

Then the polynomial

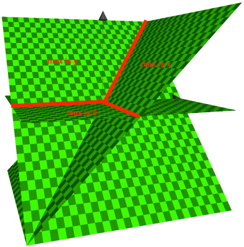
$$P_1(x, y) = \bigoplus_{(k,l)\in\Lambda_p} a_{k,l} \odot x^{\odot k} \odot y^{\odot l}$$

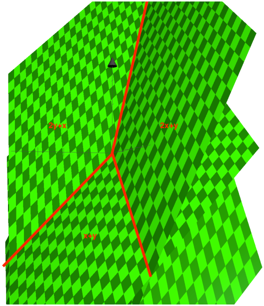
and

$$P_2(x, y) = \bigoplus_{(k,l)\in\Lambda_p} a_{k,l} \odot x^{\odot (k+i)} \odot y^{\odot (l+j)}$$

define the same tropical curve. [I08]

Introduction to tropical mathematics





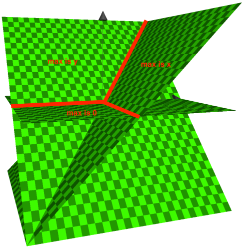
Introduction to tropical mathematics

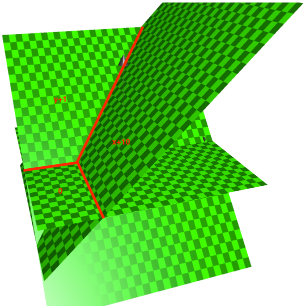
Let $P_1(x, y) = \bigoplus_{(k,l)\in\Lambda_p} a_{k,l} \odot x^{\odot k} \odot y^{\odot l}$ be tropical polynomial and C its tropical curve. The tropical curve defined by the tropical polynomial

$$P_2(x, y) = \bigoplus_{(k,l)\in\Lambda_p} (a_{k,l} + \alpha k + \beta l + \gamma) \odot x^{\odot k} \odot y^{\odot l}$$

is obtained from C by the translation $(-\alpha, -\beta)$. [I08]

Introduction to tropical mathematics





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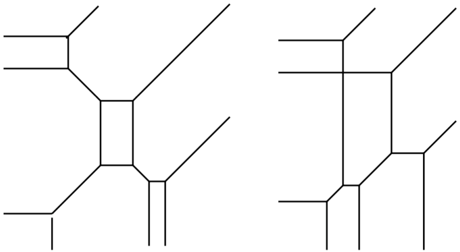
Examples of tropical curves of degree 3:



Introduction to tropical mathematics

The tropical semi-field arises naturally as the limit of the classical semi-field $(\mathbb{R}_+, +, \times)$ (Victor Maslov's dequantisation of the real numbers). [BS14, I08]

If t is a strictly positive real number, the logarithm of base t provides a bijection between $\mathbb{R}$ and $\mathbb{R}_{max} = \mathbb{R} \cup \{-\infty\}$.

This bijection induces a semi-field structure on $\mathbb{R}_{max}$ with the operations:

- $"x +_t y" = log_t(t^x + t^y)$
- $"x \times_t y" = log_t(t^x t^y) = x + y$

Introduction to tropical mathematics

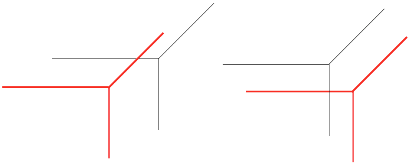
Classical addition corresponds to an exotic kind of multiplication on $\mathbb{R}_{max}$. If we let t tend to infinity, the operation $"+_t"$ tends to the tropical addition $"+"$.

By construction, $(\mathbb{R}_{max}, "+_t", "\times_t") \cong (\mathbb{R}_+, +, \times)$.

Introduction to tropical mathematics

Many ideas for the classical world generalize to the tropical world:

- Two general lines meet in a point



- Two general points lie on a unique line, five general points lie on a unique quadric, etc...
- Tropical analogues of the rank of a matrix, the independence of vectors, the determinant of a matrix, an analog of Gauss triangular form

Introduction to tropical mathematics

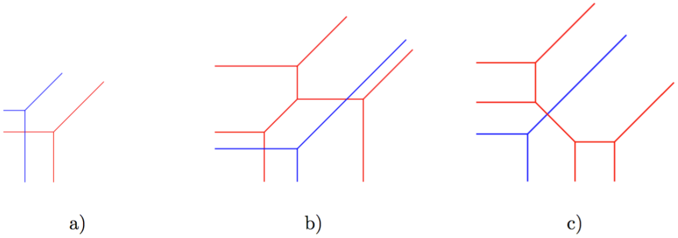
Main interests of tropical geometry: a simple model of algebraic geometry. For instance, the basic theorems from intersection theory of tropical curves require much less mathematical background than their classical counterparts.

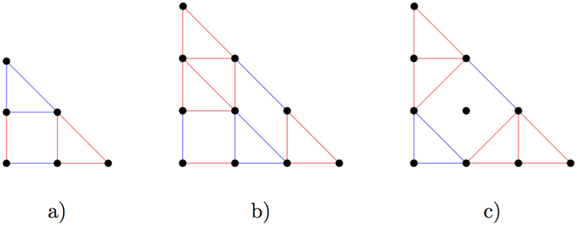
Bézout theorem:

Bézout theorem states, in classical algebraic geometry, that two algebraic curves in the plane of degrees d_1 and d_2 respectively, intersect in $d_1 d_2$ points. This theorem is also true in tropical settings.

Introduction to tropical mathematics

Bézout theorem:





Grigoriev and Podolskii's tropical Nullstellensatz

Definition

$\overrightarrow{a} \in K^n$ is a root of a polynomial f if the maximum is either attained on at least two different monomials, or is infinite.

Grigoriev and Podolskii's tropical Nullstellensatz

The tropical linear system

$$max_{1\leq j\leq n}\{a_{ij}+x_j\}\ ,\ 1\leq i\leq m$$

can be naturally associated with its matrix $A\in\mathbb{R}_{max}^{m\times n}$. We will also use a matrix notation $A\odot\overrightarrow{x}$ for such systems.

Grigoriev and Podolskii's tropical Nullstellensatz

Definition

(Macaulay matrix)

- For a system of tropical polynomials $F = \{f_1, ..., f_k\}$ in n variables, consider all the polynomials of the form $f_j \odot \overrightarrow{x}^J$, where $\overrightarrow{x}^j$ is a tropical monomial.
- Put the coefficients of these polynomials in the rows of the matrix, while the columns of the matrix correspond to monomials.
- Empty entries of the matrix we fill with $-\infty$.
- The resulting matrix we denote by M.

Grigoriev and Podolskii's tropical Nullstellensatz

More specifically, the columns of M correspond to non-negative integer vectors $I \in \mathbb{Z}_+^n$ and the rows of M correspond to the pairs (j, J), where $1 \leq j \leq k$ and $J \in \mathbb{Z}_+^n$.

For a given I and (j, J) we let the entry $m_{(j,J),I}$ be equal to the coefficient of the monomial $\overrightarrow{x}^I$ in the polynomial $\overrightarrow{x}^J \odot f_j$ (if there is no such monomial in the polynomial we assume that the entry is equal to $-\infty$).

Grigoriev and Podolskii's tropical Nullstellensatz

By M_N we denote the finite submatrix of the matrix M consisting of the columns I such that $|I| = i_1 + ... + i_n = N$ and the rows that have all their finite entries in these columns. [GP18]

Grigoriev and Podolskii's tropical Nullstellensatz

Theorem

(Tropical Dual Nullstellensatz) Consider a system of tropical polynomials $F = \{f_1, ..., f_k\}$ *in* n *variables. Denote by* d_i *the degree of the polynomial* f_i *and let* $d = max_i d_i$*. Then over the semiring* $\mathbb{R}$ *the system* F *has a root if and only if the Macauley tropical linear system* $M_N \odot \overrightarrow{y}$ *for* $N = (n+2)(\sum_{j=1}^{k} d_j)$ *has a solution. [GP18]*

Grigoriev and Podolskii's tropical Nullstellensatz

Theorem

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Grigoriev and Podolskii's tropical Nullstellensatz

Proof:

F has a root $\overrightarrow{a} \in \mathbb{R}^n \Rightarrow$ there is a solution to $M_N \odot \overrightarrow{y}$ is easy.

The coordinates y_I of $\overrightarrow{y}$ correspond to monomials $\overrightarrow{x}^I$, so let $y_I = \overrightarrow{a}^I$. Since each row of M_N correspond to the polynomial of the form $\overrightarrow{x}^J \odot f_j(\overrightarrow{x})$ and $\overrightarrow{a}$ is a root of any such polynomial, $\overrightarrow{y}$ satisfies all rows of M_N.

Grigoriev and Podolskii's tropical Nullstellensatz

The other implication is more complicated. Idea: use Carathéodory's theorem.

Theorem

(Carathéodory) If a point x of $\mathbb{R}^d$ lies in the convex hull of a set P, then there is a subset P' of P consisting of $d+1$ or fewer points such that x lies in the convex hull of P').

Mean payoff games

Problems we need to consider:

- Is a family of vectors tropically dependent? Given $m \geq n$ and an $m \times n$ matrix $A = (A_{ij})$ with entries in $\mathbb{R} \cup \{\infty\}$, are the columns of A tropically linearly dependent? I.e., can we find scalars $x_1, ..., x_n \in \mathbb{R} \cup \{-\infty\}$, not all equal to $-\infty$, such that the equation "$Ax = 0$" holds in the tropical sense, meaning that for every value of $i \in [m]$, when evaluating the expression $max(A_{ij} + x_j)j \in [n]$ the maximum is attained by at least two values of j?

Mean payoff games

- Is a tropical polyhedral cone non-trivial? Given $m \times n$ matrices $A = (A_{ij})$ and $B = (B_{ij})$ with entries in $\mathbb{R} \cup \{-\infty\}$, does there exist a vector $x \in (\mathbb{R} \cup \{-\infty\})^n$, non-identically $-\infty$ such that the inequality "$Ax \leq Bx$" holds?
- Is a tropical polyhedron empty? Given $m \times n$ matrices $A = (A_{ij})$ and $B = (B_{ij})$ with entries in $\mathbb{R} \cup \{-\infty\}$, and two vectors c, d of dimension m with entries in $\mathbb{R} \cup \{-\infty\}$, does there exist a vector $x \in (\mathbb{R} \cup \{-\infty\})^n$ such that the inequality "$Ax + c \leq Bx + d$" holds?

Mean payoff games

We look for a non-trivial element of the cone, i.e., for a solution $x = (x_j) \in \mathbb{R}_{max}$ of the system "$Ax \leq Bx$", not identically $-\infty$.

Mean payoff games

The representation of a tropical polyhedral cone by inequalities turns out to be equivalent to the description of a mean payoff game by a bipartite directed graph in which the weights indicate the payments (the weighted graph is coded by the matrices A and B).

Mean payoff games

- Define a zero-sum game
- Two players, "Max" and "Min". They alternate their move
- The state space consists of the disjoint union of the set I and the set $[n]$

Mean payoff games

Example: The system:

$$\begin{aligned} 4 + x_1 &\leq 3 + x_1 \\ max(14 + x_1, 15 + x_2) &\leq max(9 + x_1, 11 + x_2) \\ 16 + x_2 &\leq max(13 + x_1, 1 + x_2) \end{aligned}$$

can be represented using the matrices

$$A = \begin{pmatrix} 4 & -\infty \\ 14 & 15 \\ -\infty & 16 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 3 & -\infty \\ 9 & 11 \\ 13 & 1 \end{pmatrix}$$

Mean payoff games

- Row: state in which Max plays (squares)
- Column: state in which Min plays (circles)

Mean payoff games

- When the current position is $i \in I$, Player Max must choose the next state $j \in [n]$ in such a way that B_{ij} is finite.
- When the current state is $j \in [n]$, Player Min must choose the next state $i \in I$ in such a way that A_{ij} is finite.

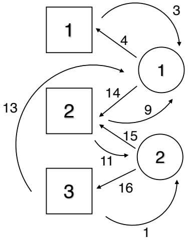
Mean payoff games

From the matrices

$$A = \begin{pmatrix} 4 & -\infty \\ 14 & 15 \\ -\infty & 16 \end{pmatrix}$$

$$B = \begin{pmatrix} 3 & -\infty \\ 9 & 11 \\ 13 & 1 \end{pmatrix}$$

we get the game:



Mean payoff games

Definition

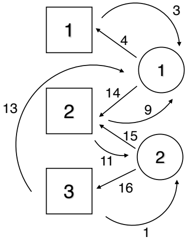
We consider the "mean payoff" game, in which the payoff of an infinite trajectory is defined as the average payment per turn received by player Max.

More later...

Mean payoff games

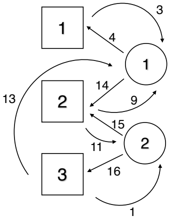
For instance, say that when in circle state 1, Min moves the token to square state 1:

- Min receives a payment of 4
- Max has to move the token on column node 1
- Max gets a a payment of 3 from Min
- The mean payoff per turn for Player Max is $-4+3=-1$



Mean payoff games

v_i^k value of MAX, initial state (i, MIN).



$$v_1^k = min(-4 + 3 + v_1^{k-1}, -14 + max(9 + v_1^{k-1}, 11 + v_2^{k-1}))$$

$$v_2^k = min(a, b)$$

$$a = -15 + max(11 + v_2^{k-1}, 9 + v_1^{k-1})$$

$$b = -16 + max(1 + v_2^{k-1}, 13 + v_1^{k-1})$$

Mean payoff games

We can also show that whether a tropical polyhedron

$$\{x | max(max_{j\in[n]}(A_{ij}+x_j), c_i) \leq max_{j\in[n]}(max(B_{ij}+x_j), d_i), i \in [m]\}$$

is non-empty reduces to whether a specific state of a mean payoff game is winning.

Mean payoff games

How do we solve the game analytically?

The residuated operator $A^\sharp$ from $(\mathbb{R} \cup \{\pm\infty\})^I$ to $(\mathbb{R} \cup \{\pm\infty\})^n$ is defined by:

$$(A^\sharp y)_j = inf_{i \in I}(-A_{ij} + y_i)$$

with the convention $(+\infty) + (-\infty) = +\infty$.

"$Ax \leq Bx \Leftrightarrow x \leq T(x)$" with $T(x) =$"$A^\sharp Bx$" ie:

$$(T(x))_j = inf_{i \in I}(-A_{ij} + max_{k \in [n]}(B_{ik} + x_k))$$

We have:

$$T : (\mathbb{R} \cup \{-\infty\})^n \to (\mathbb{R} \cup \{\pm\infty\})^n$$

Mean payoff games

- T is order preserving: $x \leq y \Rightarrow T(x) \leq f(y)$
- T is additively homogeneous: $T(a + x) = a + T(x)$
- T is the called dynamic programming operator (or Shapley operator) of a zero-sum two player deterministic game
- the states and actions are in I and $[n]$
- Reminder: Min plays in states $j \in [n]$, choose a state $i \in I$ and receive A_{ij} from Max
- Reminder: Max plays in states $i \in I$, choose a state $j \in [n]$ and receive B_{ij} from Min

Mean payoff games

Theorem

(Shapley) The vector of values v^N of the game after N turns (i.e, in finite horizon):

$$v^N = T(v^{N-1}) \ , \ v^0 = 0$$

Mean payoff games

- $[T^N(0)]_i$ is the value of the original game in horizon N with initial state i.
- $[T^N(u)]_i$ is the value of a modified game in horizon N with initial state i, in which MAX receives an additional payment of u_j in the terminal state j.

Mean payoff games

Theorem

(Bewley, Kohlberg) The mean payoff vector $lim_{N\to\infty} T^N(0)/N$ *does exist if* $T : \mathbb{R}^n \to \mathbb{R}^n$ *is semi-algebraic and nonexpansive in any norm.*

Mean payoff games

Theorem

(Akian, Gaubert, Guterman) Let T be the Shapley operator of a deterministic game. The following are equivalent.

- *initial state j is winning, meaning that* $0 \leq lim_{N\to\infty}[T^N(0)]_j/N$
- *there exists* $u \in (\mathbb{R} \cup \{-\infty\})^n$, $u_j \neq -\infty$, *and* $u \leq T(u)$

Mean payoff games

Proof idea:

- Assume that $u \in \mathbb{R}^n$ is such that $u \leq T(u)$.
- $u \leq T(u) \leq T^2(u) \leq \dots$
- $u/k \leq T^N(u)/N$
- $0 \leq lim_{N\to\infty} T^N(u)/N$
- $||T^N(u) - T^N(0)||_\infty \leq ||u - 0||_\infty = ||u||_\infty$
- $0 \leq lim_{N\to\infty} T^N(0)/N$
- all states are winning.

The converse follows from a fixed point theorem of Kohlberg (a nonexpansive piecewise linear map has an invariant half-line).

Mean payoff games

Theorem

Every order preserving and additively homogeneous map $g : \mathbb{R}^n \to \mathbb{R}^n$ can be written as the dynamic programming operator of a zero-sum two player deterministic game :

$$[g(x)]_j = inf_{i \in I} max_{k \in [n]} (r_{jik} + x_k)$$

Every dynamic programming operator g as above can be written as $g(x) = A^\sharp Bx$ for some (infinite) matrices $A, B \in R^{I' \times [n]}$
Thus $C := \{x \in (\mathbb{R} \cup \{-\infty\})^n | x \leq g(x)\}$is a tropical convex cone.

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