Introduction to tropical geometry

Toward a tropical Nullstellensatz

Raphaël Pellegrin
under the supervision of Marianne Akian and Stéphane Gaubert

École Polytechnique & the French Institute for Research in Computer Science and Automation (INRIA)

October 10, 2018

Overview

Introduction to tropical geometry

The tropical semi-field

Tropical polynomials

Grigoriev and Podolskii's tropical Nullstellensatz

Mean payoff games

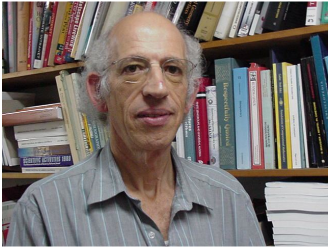
References

Introduction to tropical mathematics

Tropical?

Introduction to tropical mathematics

The foundations of tropical mathematics were laid in the 1960s by Cuninghame-Green and Vorobyev. They were named in honour of Imre Simon who lived and worked in Sao Paulo.



Introduction to tropical mathematics

In tropical geometry, we consider the max-plus semi-field $\mathbb{R}_{max}$, the set $\mathbb{R} \cup \{-\infty\}$ with the two operations, $x \oplus y = max\{x, y\}$ and $x \odot y = x + y$.

For instance $3 \oplus 5 = 5$, $3 \odot 5 = 3 + 5 = 8$, $2^{\odot 3} = 2 \times 3 = 6$.

We sometimes use quotation marks to denote operations in the tropical world: "$3 + 3 = 3$", "$\sqrt{-1} = -0.5$"

Introduction to tropical mathematics

We denote $\overrightarrow{x} = (x_1, ..., x_n)$ and for $I = (i_1, ..., i_n)$ we introduce the notation $\overrightarrow{x}^I = x_1^{\odot i_1} \odot ... \odot x_n^{\odot i_n} = i_1 x_1 + ... + i_n x_n$.

Tropical monomial in variables $\overrightarrow{x} = (x_1, ..., x_n)$:
$m(\overrightarrow{x}) = c \odot x_1^{\odot i_1} \odot ... \odot x_n^{\odot i_n}$ where all exponents are non-negative integers. In classical terms, $m(\overrightarrow{x}) = c + \langle i, x \rangle$ is an affine function with non-negative integer slope.

Tropical polynomial: $"\sum_{i=1}^{n} M_i(\overrightarrow{x})" = max_i M_i(\overrightarrow{x})$
each $M_i(\overrightarrow{x})$ is a tropical monomial in variables $\overrightarrow{x} = (x_1, ..., x_n)$

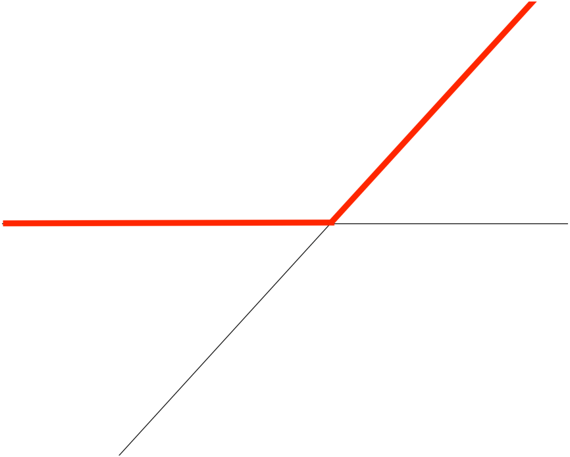
Introduction to tropical mathematics

Definition

The degree of a tropical monomial is the sum of its exponents. The degree of a tropical polynomial f, $deg(f)$, is the maximal degree of its monomials.

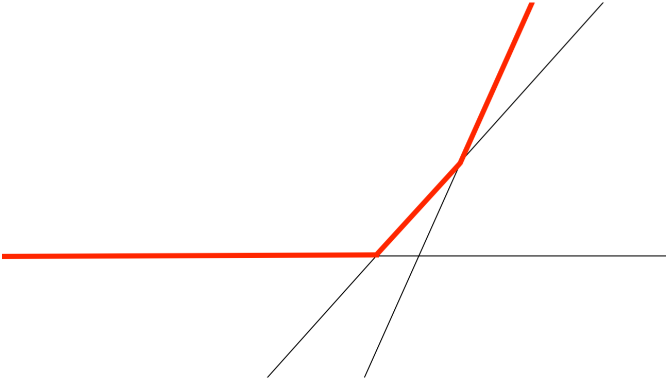
Introduction to tropical mathematics

In one variable: let us look at $P_1(x) = "0 + x"$



Introduction to tropical mathematics

... and at $P_2(x) =" 0 + x + (-1)x^2" =" (-1)(x+0)(x+1)"$



Introduction to tropical mathematics

In one variable: Tropical roots of the polynomial $P(x)$: points x_0 at which the graph $P(x)$ has a corner at x_0.

The tropical semi-field is algebraically closed. In other words every tropical polynomial of degree d has exactly d roots when counted with multiplicities.

Introduction to tropical mathematics

In two variables: $P(x, y) =" \sum_{i,j} a_{i,j} x^i y^j " = max_{i,j}(a_{i,j} + ix + jy)$.

Introduction to tropical mathematics

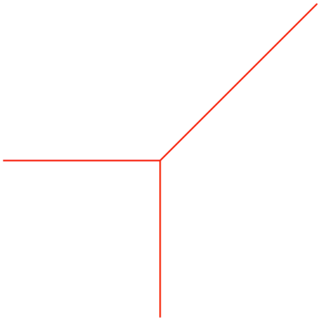
The tropical curve C defined by $P(x, y)$ is defined as the corner locus of this function (ie all the points for which the maximum of $P(x, y)$ is attained at least twice).

$P(x, y) ='' \sum_{i,j} a_{i,j} x^i y^j '' = max_{i,j}(a_{i,j} + ix + jy)$

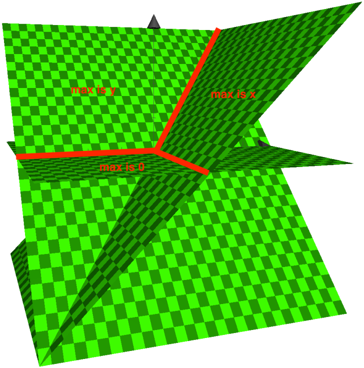
C is the set of points (x_0, y_0) of $\mathbb{R}^2$ such that there exists pairs $(i, j) \neq (k, l)$ satisfying $P(x_0, y_0) = a_{i,j} + ix_0 + jy_0 = a_{k,l} + kx_0 + ly_0$

Introduction to tropical mathematics

Example: $P(x,y) = "0 + x + y" = max(0,x,y)$. We look at $x = 0 \geq y$, $y = 0 \geq x$, $x = y \geq 0$:



Introduction to tropical mathematics



Introduction to tropical mathematics

The tropical semi-field arises naturally as the limit of the classical semi-field $(\mathbb{R}_+, +, \times)$ (Victor Maslov's dequantisation of the real numbers). [BS14, I08]

This bijection induces a semi-field structure on $\mathbb{R}_{max}$ with the operations:

- $"x +_t y" = log_t(t^x + t^y)$
- $"x \times_t y" = log_t(t^x t^y) = x + y$

Introduction to tropical mathematics

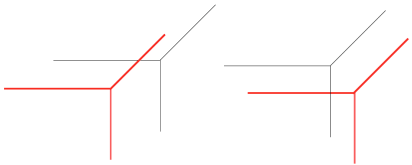
- $"x +_t y" = log_t(t^x + t^y)$
- $"x \times_t y" = log_t(t^x t^y) = x + y$

- Classical addition corresponds to an exotic kind of multiplication on $\mathbb{R}_{max}$.
- If we let t tend to infinity, the operation $"+_t"$ tends to the tropical addition $"+"$.

Introduction to tropical mathematics

Many ideas for the classical world generalize to the tropical world:

- Two lines meet in a point



- Bézout theorem
- Two general points lie on a unique line, five general points lie on a unique quadric, etc...

Grigoriev and Podolskii's tropical Nullstellensatz

Definition

$\overrightarrow{a} \in K^n$ is a root of a polynomial f if the maximum is either attained on at least two different monomials, or is infinite.

Grigoriev and Podolskii's tropical Nullstellensatz

The tropical homogeneous linear system

$$max_{1 \leq j \leq n}\{a_{ij} + x_j\} \ , 1 \leq i \leq m$$

can be naturally associated with its matrix $A \in \mathbb{R}_{max}^{m \times n}$. We will also use a matrix notation $A \odot \overrightarrow{x}$ for such systems.

Grigoriev and Podolskii's tropical Nullstellensatz

Theorem

(Tropical Dual Nullstellensatz) Consider a system of tropical polynomials $F = \{f_1, ..., f_k\}$ *in* n *variables. Denote by* d_i *the degree of the polynomial* f_i *and let* $d = max_i d_i$*. Then over the semiring* $\mathbb{R}$ *the system* F *has a root if and only if the Macauley tropical linear system* $M_N \odot \overrightarrow{y}$ *for* $N = (n+2)(\sum_{j=1}^{k} d_j)$ *has a solution. [GP18]*

Mean payoff games

- Is a family of vectors tropically dependent?

  I.e: Given $m \geq n$ and an $m \times n$ matrix $A = (A_{ij})$ with entries in $\mathbb{R} \cup \{\infty\}$, are the columns of A tropically linearly dependent?

  I.e., can we find scalars $x_1, ..., x_n \in \mathbb{R} \cup \{-\infty\}$, not all equal to $-\infty$, such that the equation "$Ax = 0$" holds in the tropical sense, meaning that for every value of $i \in [m]$, when evaluating the expression $max(A_{ij} + x_j) j \in [n]$ the maximum is attained by at least two values of j?

  We use Mean Payoff Games (MPG).

References

AGG12 [AGG12] M. Akian, S. Gaubert, and A. Guterman. Tropical polyhedra are equivalent to mean payoff games. International Journal of Algebra and Computation, 22(1):125001 (43 pages), 2012.

AGK10 [AGK10] X. Allamigeon, S. Gaubert, and R. Katz. Tropical polar cones, hypergraph transversals, and mean payoff games. Linear Algebra and its Applications, 435(7):1549-1574, 2011.

BS14 [BS14] E. Brugallé, K. Shaw. A bit of tropical geometry. The American Mathematical Monthly Volume 121, Number 7, pp 563-589, 2014.

GP18 [GP18] D. Grigoriev and V. V. Podolskii, Tropical Effective Primary and Dual Nullstellensätze, Discrete & Computational Geometry, Volume 59, Issue 3, pp 507-552, 2018.

I08 [I08] I. Itenberg, Introduction à la géometrie tropicale, Available from: http://www.math.polytechnique.fr/xups/xups08-01.pdf [Accessed 18th July 2018]